Conservation of circulation in magnetohydrodynamics

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We demonstrate at both the Newtonian and (general) relativistic levels the existence of a generalization of Kelvin's circulation theorem (for pure fluids) that is applicable to perfect magnetohydrodynamics. The argument is based on the least action principle for magnetohydrodynamic flow. Examples of the new conservation law are furnished. The new theorem should be helpful in identifying new kinds of vortex phenomena distinct from magnetic ropes or fluid vortices.

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I. INTRODUCTION

Kelvin's theorem on the conservation of circulation of a simple perfect fluid has played an important role in the development of hydrodynamics. For instance, it shows that potential flows are possible, that isolated vortices can exist, that they obey the Helmholtz laws, etc. Kelvin's theorem is valid only for flows in which the body force per unit mass is a gradient; mostly this includes incompressible or isentropic flows of one-component fluids.

Most flows in geophysics and astrophysics are more complicated. In particular, many fluids in the real world carry magnetic fields: they are magnetofluids. Yet the Lorentz force per unit mass on a magnetofluid is almost *never* a perfect gradient. Thus the circulation theorem in its original form is almost never true in magnetohydrodynamics (MHD). Must we then surrender the many insights that Kelvin's theorem conferred on pure hydrodynamics?

Not necessarily. One might speculate that a suitable combination of fluid velocity $\mathbf{v}$ and magnetic induction $\mathbf{B}$ may inherit the property of having a ''circulation'' on a closed curve which is preserved as that curve is dragged with the magnetofluid. Such conserved circulation might play as useful a role in MHD as has Kelvin's circulation in pure fluid dynamics. For example, it might help characterize a set of magnetoflows as being potential in some sense, with consequent simplification of this intricate subject. Or it might help to characterize a new type of vortex, a hybrid vorticity–magnetic rope. In view of the importance of the vortex phenomenon in contemporary physics, this last possibility is by itself ample reason to delve into the subject.

Two decades ago, Bekenstein and E. Oron [1] discovered, with the formalism of relativistic perfect MHD, a circulation theorem of the above kind. Although some of its consequences for new helicity conservation laws have been explored [2], this new conserved circulation has remained obscure. Contributing to this, no doubt, is the fact that it has only been derived relativistically, and that this derivation is an intricate one, even for relativistic MHD. In addition, Oron's derivation assumes both stationary symmetry and axisymmetry, while it is well known that Kelvin's theorem requires neither of these.

In the present paper we use the least action principle to give a rather straightforward existence proof for a generically conserved hybrid velocity–magnetic field circulation within the framework of perfect MHD which does not depend on spacetime symmetries. We do this at both the Newtonian (Sec. II) and general relativistic (Sec. III) levels; the importance of MHD effects in pulsars, active galactic nuclei, and cosmology underscores that this last arena is not just of academic importance.

As mentioned, we approach the whole problem not from equations of motion, but from the least action principle. Lagrangians for nonrelativistic pure perfect flow have been proposed by Herivel [3], Eckart [4], Lin [5], Seliger and Witham [6], Mittag, Stephen, and Yourgrau [7], and others. Many of the proposed Lagrangians necessarily imply irrotational flow, i.e., not generic flow, a deficiency that is often missed by the authors. Lin [5] introduced a device that allows vortical flows to be encompassed. This device was used by Seliger and Witham. Lagrangians for nonrelativistic perfect MHD flow in Eulerian coordinates have been proposed by Eckart [8], Henyey [9], Newcomb [10], Lundgren [11], and others.

In special relativity Penfield [12] proposed a perfect fluid Lagrangian that admits vortical isentropic flow. The early general relativistic Lagrangian of Taub [13,14] as well as the more recent one by Kodama *et al.* [15] describe only irrotational perfect fluid flows. The Lin device is incorporated by Schutz [16], whose perfect fluid Lagrangian admits vortical as well as irrotational flows in general relativity. Carter [17,18] introduced Lagrangians for particlelike motions from which can be inferred the properties of fluid flows, including vortical ones. Achterberg [19] proposed a general relativistic MHD action, which, however, describes only ''irrotational'' flows. Thompson [20] used this Lagrangian in the extreme relativistic limit. Heyl and Hernquist [21] modified it to include QED effects. In this paper we follow mostly Seliger and Witham [6] and Schutz [16].

In Sec. II A we propose a nonrelativistic MHD Lagrangian, and show in Secs. II B and II C that it gives rise to the correct equations of motion for the density, entropy, velocity, and magnetic fields in Newtonian MHD. In Sec. II D we derive from it the conserved circulation, defined in terms of a new vector field $\mathbf{R}$, and discuss its invariance under redefi-

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nition of **R**. Section II E furnishes two examples of the conserved circulation in action. In Sec. III A we collect all the equations of motion of general relativistic MHD, and propose a general relativistic MHD Lagrangian in Sec. III B. Sections III C and III D recover all the relativistic MHD equations of motion from it. Finally, in Sec. III E we generalize the conserved MHD circulation to the general relativistic case.

II. VARIATIONAL PRINCIPLE IN EULERIAN COORDINATES

A. The Lagrangian density

Perfect MHD describes situations where the flow is nondissipative, and, in particular, when the magnetoflow has ''infinite conductivity,'' and where Maxwell's displacement current may be neglected in Ampère's equation. We shall adopt this approximation. We work in Eulerian coordinates: all physical quantities are functions of coordinates x_i or $\mathbf{r}$ that describe a fixed point in space. We first summarize the MHD equations. We work in units for which $c=1$.

First of all, the fluid obeys the equation of continuity ($\partial_t \equiv \partial/\partial t$)

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{2.1}$$

where $\rho(\mathbf{r},t)$ is the mass density per unit volume of the fluid and $\mathbf{v}(\mathbf{r},t)$ is the fluid's velocity field. Second, since there is no dissipation, s, the entropy per unit mass, must be conserved along the flow:

$$Ds \equiv \partial_t s + \mathbf{v}\cdot\nabla s = 0. \tag{2.2}$$

Here we have defined the convective derivative D, which in Cartesian coordinates has the same form for scalars or vectors. With the help of Eq. (2.1) this equation can be written as

$$\partial_t(\rho s) + \nabla\cdot(\rho s\mathbf{v}) = 0. \tag{2.3}$$

Third, ''infinite conductivity'' implies that $\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}=0$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively. Combining this with Faraday's equation yields the so-called field-freezing equation

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{2.4}$$

which implies Alfvén's law of conservation of the magnetic flux through a closed loop moving with the flow. Finally, the evolution of the velocity field is governed by the MHD Euler equation,

$$\rho D\mathbf{v} = -\nabla p - \rho\nabla U + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}, \tag{2.5}$$

where p is the fluid's pressure (here assumed isotropic), and $U(\mathbf{r},t)$ is the gravitational potential.

The least action principle is in general

$$\delta S[f_a] \equiv \delta\int dt\int d^3r\,\mathcal{L}(f_a,\partial_t f_a,\nabla f_a) = 0. \tag{2.6}$$

Here the action S is a functional of various fields $f_a(\mathbf{r},t)$, $a=1,2,\ldots$. One varies each f_a, transfers time and space derivatives of each variation δf_a to the adjacent factor by integration by parts, and sets to zero the overall coefficient of the bare δf_a. This gives us the Lagrange-Euler equation

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t f_a)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla f_a}\right) - \frac{\partial\mathcal{L}}{\partial f_a} = 0. \tag{2.7}$$

It is usually more convenient to get the equation for each f_a *ab initio* by the above procedure, rather than by using Eq. (2.7).

We now propose the following Lagrangian *density* for MHD flow of a perfect infinitely conducting fluid which incorporates Eqs. (2.1)–(2.4), as three Lagrange constraints

$$\begin{aligned}\mathcal{L} = {}& \rho\mathbf{v}^2/2 - \rho\epsilon(\rho,s) - \rho U - \mathbf{B}^2/(8\pi) + \phi[\partial_t\rho + \nabla\cdot[(\rho\mathbf{v})]\\ &+ \eta[\partial_t(\rho s) + \nabla\cdot(\rho s\mathbf{v})] + \lambda[\partial_t(\rho\gamma) + \nabla\cdot(\rho\gamma\mathbf{v})]\\ &+ \mathbf{K}\cdot[\partial_t\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B})].\end{aligned} \tag{2.8}$$

In the above, $\epsilon(\rho,s)$ is the thermodynamic internal energy per unit mass; in the total Lagrangian the corresponding total internal energy enters as a potential energy. The magnetic energy, the volume integral of $\mathbf{B}^2/(8\pi)$, also enters the total Lagrangian as a potential energy.

In Eq. (2.8) ϕ,η are Lagrange multiplier fields that locally enforce the conservation laws (2.1), (2.2), as may be verified by varying with respect to these multipliers. $\mathbf{K}$ is a triplet of Lagrange multiplier fields that enforce the field-freezing constraint Eq. (2.4): varying with respect to $\mathbf{K}$ reproduces Eq. (2.4) at every point and time. Finally, λ is a Lagrange multiplier field that enforces the Lin constraint on a new field γ:

$$\partial_t(\rho\gamma) + \nabla\cdot(\rho\mathbf{v}\gamma) = 0 \quad \text{or} \quad D\gamma = 0. \tag{2.9}$$

Here we have used Eq. (2.1) to reduce to the second form. Lin's field γ, like s, is conserved along the flow, but unlike s it does not occur elsewhere in the Lagrangian. Lin interprets $\gamma(\mathbf{r},t)$ as one of the three initial *Lagrangian* coordinates that label each fluid element. But whatever the interpretation, the condition (2.9) is essential so that the flow can be vortical also in the limit $\mathbf{B}\to\mathbf{0}$. This matter is further discussed in the following section.

B. The equations of motion

Can our proposed Lagrangian density reproduce all the equations of motion of perfect MHD flow? We have already seen that it does reproduce Eqs. (2.1), (2.2), and (2.4). Let us now vary γ to get

$$D\lambda = 0, \tag{2.10}$$

so that λ, like γ, is conserved with the flow. Both this and Eq. (2.9) will be essential in demonstrating the existence of the new conserved circulation. Next we vary s; remembering that $(\partial\epsilon/\partial s)_\rho$ is just the fluid's temperature T, we have

$$D\eta = -T, \tag{2.11}$$

which establishes that η decreases along the flow. The next variation is one with respect to ρ. Recalling that $(\partial\epsilon/\partial\rho)_s = p/\rho^2$, introducing the enthalpy per unit mass $w=\epsilon+p/\rho$, and using Eqs. (2.10) and (2.11), we get

$$D\phi = v^2/2 - w + T - U. \tag{2.12}$$

When we vary $\mathbf{v}$ in the action we may take advantage of the identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$ and Gauss's theorem to flip the curl operation from $\delta\mathbf{v}\times\mathbf{B}$ onto $\mathbf{K}$. Then the identity $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = -\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}$ helps to shift the $\delta\mathbf{v}$ into the position of a factor in a scalar product. We may then factor out the common $\delta\mathbf{v}$ and isolate the vector equation

$$\mathbf{v} = \nabla\phi + \gamma\nabla\lambda + s\nabla\eta + \mathbf{Q}, \tag{2.13}$$

$$\mathbf{Q} \equiv \mathbf{B}\times\mathbf{R}/\rho, \tag{2.14}$$

where $\mathbf{R}\equiv\nabla\times\mathbf{K}$. This is neither a solution for $\mathbf{v}$ (λ and η not known), nor an equation of motion ($\mathbf{v}$ appears undifferentiated). In the next subsection we show that this prescription for $\mathbf{v}$ leads to the MHD Euler equation (2.5).

Expression (2.13) shows the importance of including Lin's field γ. For suppose we consider an unmagnetized fluid in isentropic ($s=\text{const}$) flow. Without γ the expression for $\mathbf{v}$ is a perfect gradient, which means the proposed Lagrangian density describes only irrotational flows, a small subset of all possible ones. It is well known [6,7] that this problem does not appear when one couches the problem in Lagrangian coordinates because one gets then an equation, not for $\mathbf{v}$, but for the fluid's acceleration. Lin's [5] way out of this difficulty is to remember that the initial coordinates of the fluid element are maintained throughout its flow. These coordinates ''label'' the element, and this can be interpreted as a triplet of constraints (one for each coordinate) of the form $\lambda_i(\partial b_i/\partial t + \nabla b_i)$, where $\mathbf{b}$ is the initial vector coordinate for the element in question. Lundgren [11] used this triplet form for the MHD case. It was later shown (see, for example, [6]) that the triplet can be reduced to a single constraint with the help of Pfaff's theorem. One thus returns to form (2.8) of the Lagrangian density and Eq. (2.13) for the fluid velocity. The vorticity is now (still excluding $\mathbf{B}$)

$$\boldsymbol{\omega} = \nabla\times\mathbf{v} = \nabla\gamma\times\nabla\lambda + \nabla s\times\nabla\eta, \tag{2.15}$$

so we see that isentropic vortical flow is possible.

In the MHD case, the magnetic term in Eq. (2.13) contributes to the vorticity. Henyey [9], who suggested a Lagrangian density similar to ours, occasionally dropped the Lin term in the MHD case. However, we shall retain the Lin term throughout. It might seem peculiar at first that adding a constraint like Lin's permits the appearance of solutions (vortical) that were forbidden before it was imposed. But we must remember that we add to the Lagrangian not only a constraint, but also a new degree of freedom $\gamma(\mathbf{r},t)$, and it is natural that with more degrees of freedom the class of allowed flows will expand.

Finally, we vary $\mathbf{B}$ in the action; by similar manipulation to those that gave Eq. (2.13) we get

$$\partial_t\mathbf{K} = \mathbf{v}\times\mathbf{R} - \mathbf{B}/(4\pi). \tag{2.16}$$

Taking the curl of this equation we get the more convenient one

$$\partial_t\mathbf{R} = \nabla\times[\mathbf{v}\times\mathbf{R} - \mathbf{B}/(4\pi)] = \nabla\times(\mathbf{v}\times\mathbf{R}) - \mathbf{J}. \tag{2.17}$$

Here $\mathbf{J}=\nabla\times\mathbf{B}/4\pi$ is the electric current density coming from Ampère's equation. Notice the similarity between Eq. (2.17) and (2.4). Equation (2.17) says that the rate of change of the flux of $\mathbf{R}$ through the surface spanning a closed curve carried with the flow equals minus the flux of the electric current density through that curve.

C. The MHD Euler equation

We now show that the Lagrangian density (2.8) yields the correct MHD Euler equation. We first operate with the convective derivative D on Eq. (2.13) remembering that $Ds=0$ and $D\gamma=0$:

$$D\mathbf{v} = D\nabla\phi + \gamma D\nabla\lambda + sD\nabla\eta + D\mathbf{Q}. \tag{2.18}$$

We now use the identity

$$D\nabla = \nabla D - (\nabla\mathbf{v})\cdot\nabla, \tag{2.19}$$

where in Cartesian coordinates

$$[(\nabla\mathbf{v})\cdot\nabla]_i \equiv \sum_j \frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{2.20}$$

in conjunction with Eqs. (2.10)–(2.12) to transform Eq. (2.18) into

$$D\mathbf{v} = \nabla(v^2/2 - w + Ts - U) - s\nabla T - s(\nabla\mathbf{v})\cdot\nabla\eta - (\nabla\mathbf{v})\cdot\nabla\phi - \gamma(\nabla\mathbf{v})\cdot\nabla\lambda + D\mathbf{Q}. \tag{2.21}$$

From the thermodynamic identity $dw = Tds + dp/\rho$ we infer

$$-\nabla w + T\nabla s = -\nabla p/\rho, \tag{2.22}$$

and we also have $\nabla v^2/2 = (\nabla\mathbf{v})\cdot\mathbf{v}$, where the meaning of the right hand side is clear by analogy with Eq. (2.20). Thus Eq. (2.21) turns into

$$D\mathbf{v} = -\nabla p/\rho - \nabla U + (\nabla\mathbf{v})\cdot(\mathbf{v} - \nabla\phi - s\nabla\eta - \gamma\nabla\lambda) + D\mathbf{Q}. \tag{2.23}$$

Finally, comparing with Eq. (2.13) we see that the last brackets stand for $\mathbf{Q}$ so that

$$D\mathbf{v} = -\nabla p/\rho - \nabla U + (\nabla\mathbf{v})\cdot\mathbf{Q} + D\mathbf{Q}. \tag{2.24}$$

Thus, magnetic term aside, we have recovered the Euler equation (2.5).

We now go on to calculate the $\mathbf{Q}$ dependent terms. We may rewrite the equation of continuity (2.1) as

$$D\rho = -\rho\nabla\cdot\mathbf{v}. \tag{2.25}$$

With this, the Gauss law $\nabla\cdot\mathbf{B}=0$, and the identity $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{B}\nabla\cdot\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{A}\nabla\cdot\mathbf{B}$, Eq. (2.4) may be recast in the well known form

$$D(\mathbf{B}/\rho) = [(\mathbf{B}/\rho)\cdot\nabla]\mathbf{v}. \tag{2.26}$$

Analogously, because $\nabla\cdot\mathbf{R}=0$, Eq. (2.17) may be put in the form

$$D\mathbf{R}=(\mathbf{R}\cdot\nabla)\mathbf{v}-\mathbf{R}\nabla\cdot\mathbf{v}-\mathbf{J}. \tag{2.27}$$

Therefore,

$$\begin{aligned}(\nabla\mathbf{v})\cdot\mathbf{Q}+D\mathbf{Q}&=-(\nabla\mathbf{v})\cdot(\mathbf{R}\times\mathbf{B}/\rho)-D\mathbf{R}\times\mathbf{B}/\rho\\&\quad-\mathbf{R}\times D(\mathbf{B}/\rho)\\&=-(\nabla\mathbf{v})\cdot(\mathbf{R}\times\mathbf{B}/\rho)-[(\mathbf{R}\cdot\nabla)\mathbf{v}]\times(\mathbf{B}/\rho)\\&\quad+(\nabla\cdot\mathbf{v})\mathbf{R}\times(\mathbf{B}/\rho)-\mathbf{R}\times[(\mathbf{B}/\rho)\cdot\nabla]\mathbf{v}\\&\quad+\mathbf{J}\times\mathbf{B}/\rho.\end{aligned} \tag{2.28}$$

The four terms in the second version of Eq. (2.28) involving derivatives of $\mathbf{v}$ can be shown to cancel out by expanding them out in Cartesian coordinates. Hence, Eq. (2.24) is the magnetic Euler equation with the usual Lorentz force per unit mass, $\mathbf{J}\times\mathbf{B}/\rho$, in addition to the pure fluid terms. The fact that we obtain the correct MHD equations (2.1), (2.2), (2.4), and (2.5) is testament to the correctness of our proposed Lagrangian density Eq. (2.8). Note that Lin's field γ has disappeared from the final equation of motion.

D. Circulation conservation law

With the help of the above formalism, we can now prove the existence of a generalization of Kelvin's circulation theorem applicable to perfect MHD. Let us calculate the line integral of the vector

$$\mathbf{Z}=\mathbf{v}+\mathbf{R}\times\mathbf{B}/\rho \tag{2.29}$$

along a closed curve $\mathcal{C}$ drifting with the fluid:

$$\Gamma=\oint_{\mathcal{C}}\mathbf{Z}\cdot d\mathbf{r}. \tag{2.30}$$

According to Eq. (2.13) this integral is

$$\Gamma=\oint_{\mathcal{C}}\nabla\phi\cdot d\mathbf{r}+\oint_{\mathcal{C}}\gamma\nabla\lambda\cdot d\mathbf{r}+\oint_{\mathcal{C}}s\nabla\eta\cdot d\mathbf{r}. \tag{2.31}$$

The term involving ϕ obviously vanishes (we assume all the Lagrange multipliers are single valued). For like reason so does the term involving η in the isentropic ($s=$const) case as s can be taken out of the integral. The middle integral can be written $\oint_{\mathcal{C}}\gamma d\lambda$, where $d\lambda\equiv\nabla\lambda\cdot d\mathbf{r}$. But Eqs. (2.9) and (2.10) tell us that both γ and λ are conserved along the flow. Hence Γ remains constant as $\mathcal{C}$ drifts along with the flow. Since, in the limit $\mathbf{B}\to\mathbf{0}$, Γ becomes Kelvin's circulation, we have found an extension of Kelvin's theorem to perfect MHD. Obviously the conservation of Γ implies the conservation of the flux of $\nabla\times\mathbf{Z}$ through $\mathcal{C}$.

The vector field $\mathbf{R}$ is not unique for a given physical situation. For example, the change $\mathbf{R}\to\mathbf{R}+k\mathbf{B}$ (k a real constant) leaves invariant all equations of motion, Eqs. (2.9)–(2.14), (2.17), and (2.24), as well as the conserved circulation expressions (2.29) and (2.30). In addition, suppose that at time $t=0$ we define an arbitrary solenoidal (divergence-free) field $\mathbf{b}$ all over the flow, and then evolve it in time as a passive vector, i.e., in accordance with the frozen-in field equation (2.4). Comparing with Eq. (2.17) we see that $\mathbf{R}+k\mathbf{b}$ and $\mathbf{R}$ obey the same equation, and both are permanently solenoidal [this property is obviously preserved by Eqs. (2.4) and (2.17) in the MHD approximation].

If in $\mathbf{Z}$ we use $\mathbf{R}+k\mathbf{b}$ in lieu of $\mathbf{R}$ to construct the conserved circulation, Γ gets the additional contribution

$$\Delta\Gamma=k\oint_{\mathcal{C}}(\mathbf{b}\times\mathbf{B}/\rho)\cdot d\mathbf{r}=k\oint_{\mathcal{C}}\mathbf{B}\cdot(d\mathbf{r}\times\mathbf{b}/\rho). \tag{2.32}$$

Here we have used a well known vector identity. Now by analogy with $\mathbf{B}$, $\mathbf{b}$ obeys Eq. (2.26), which tells us that any two elements of the fluid permanently lie on one and the same line of $\mathbf{b}/\rho$, and their distance, if small, is proportional to $|\mathbf{b}|/\rho$ [22]. We can always make $\mathbf{b}$ small. Then $d\mathbf{r}\times\mathbf{b}/\rho$ is a vectorial element of area of a narrow closed strip carried along by the fluid, one of whose edges coincides with $\mathcal{C}$. The integral in Eq. (2.32) is just the flux of magnetic induction through this strip (not through the space bounded by the strip), and we know this is conserved by virtue of Alfvén's law.

Thus with the change $\mathbf{R}\to\mathbf{R}+k\mathbf{b}$ we added some conserved magnetic flux to Γ, and did not get a new conserved circulation. The MHD flow $\{\mathbf{B},\mathbf{v},\rho,p\}$ is evidently unchanged because the MHD Euler equation (2.5) does not contain $\mathbf{R}$, so we must conclude that in the expression for $\mathbf{v}$, Eqs. (2.13), (2.14), the change of the $\mathbf{Q}$ term must be compensated by suitable changes in the Lagrange multipliers $\phi+s\eta$ and λ (recall that we are working with $s=$const). Indeed, the initial choice of $\mathbf{b}$ involves a choice of two functions because of the $\nabla\cdot\mathbf{b}=0$ constraint, so that the two functions $\phi+s\eta$ and λ are just enough to absorb the change $\mathbf{R}\to\mathbf{R}+k\mathbf{b}$ thus generated and leave $\mathbf{v}$ unchanged. It is not possible to eliminate $\mathbf{R}$ altogether by the change $\mathbf{R}\to\mathbf{R}+k\mathbf{b}$ because $\mathbf{R}$ and $\mathbf{b}$ obey different equations. This means the circulation conservation law we have found cannot be reduced to an Alfvén type law; it is a new law.

In Sec. III E we shall discuss the freedom inherent in $\mathbf{R}$ by a covariant procedure. Fixing the freedom is a necessary step in any attempt to exhibit explicitly the conserved circulation.

E. Examples

First consider a situation where the fluid is isentropic but not flowing: $\mathbf{v}=\mathbf{0}$. It follows from Eq. (2.1) that $\rho=\rho_0(\mathbf{r})$, and from Eq. (2.4) that $\mathbf{B}=\mathbf{B}_0(\mathbf{r})$. From these facts and Eq. (2.17) we see that

$$\mathbf{R}=-t\nabla\times\mathbf{B}_0(\mathbf{r})/(4\pi)+\mathbf{R}_0(\mathbf{r}). \tag{2.33}$$

Although the physical quantities are stationary, $\mathbf{R}$ is not. This is so because, like the electromagnetic potential, $\mathbf{R}$ is not a measurable quantity, being subject to ''gauge changes'' $\mathbf{R}\to\mathbf{R}+\mathbf{b}$ as already discussed. According to Eq. (2.29) the conserved circulation (around a contour fixed in space because $\mathbf{v}=\mathbf{0}$) should be

$$\Gamma = -t \oint_{\mathcal{C}} \frac{(\nabla \times \mathbf{B}_0)\times \mathbf{B}_0}{4\pi\rho_0}\cdot d\mathbf{r} + \oint_{\mathcal{C}} \frac{\mathbf{R}_0 \times \mathbf{B}_0}{4\pi\rho_0}\cdot d\mathbf{r}. \tag{2.34}$$

On the face of it, the time dependence of the first term in this simple situation puts the claimed circulation conservation law in jeopardy. However, according to the magnetic Euler equation (2.5), the first integrand here is equal to $\nabla U + \nabla p/\rho_0$ which, by virtue of Eq. (2.19) and the isentropic nature of the fluid, is a perfect gradient (for isentropic flow $\nabla p/\rho_0 = \nabla w$). Hence the first integral vanishes, and the circulation is indeed time independent as required by our theorem.

As a second example consider an axisymmetric differentially rotating fluid exhibiting a purely poloidal magnetic field. Let the flow also be isentropic and stationary. We choose to work in cylindrical coordinates $\{\varrho,\phi,z\}$; the caret will denote a unit vector in the stated direction. It then follows that $\rho=\rho_0(\mathbf{r})$, $\mathbf{B}=\mathbf{B}_0(\mathbf{r})$, and $\mathbf{v}=\Omega\varrho\hat{\boldsymbol{\phi}}$, where $\Omega(\varrho,z)$ is the angular velocity of the fluid. It is well known [23,24] that for axisymmetric fields the curl of a poloidal field is a toroidal one, and the toroidal field has only a $\hat{\boldsymbol{\phi}}$ component. Therefore, the electric current density $\mathbf{J}=\nabla\times\mathbf{B}/(4\pi)$ is everywhere collinear with $\mathbf{v}$ and time independent. Since the problem is stationary, Ω satisfies Ferraro's [25,24] law of isorotation $\mathbf{B}\cdot\nabla\Omega=0$. In addition the field must be torque-free [24], i.e., no Lorentz force in the $\hat{\boldsymbol{\phi}}$ direction. This condition is identically satisfied for a purely poloidal field. Combining all of the above we get the following solution of Eq. (2.17):

$$\mathbf{R} = -t\mathbf{J}. \tag{2.35}$$

According to Eq. (2.29) the conserved circulation should be

$$\Gamma = \oint_{\mathcal{C}} \Omega\varrho^2 d\phi - t\oint_{\mathcal{C}} \frac{\mathbf{J}\times\mathbf{B}_0}{4\pi\rho_0}\cdot d\mathbf{r}, \tag{2.36}$$

where we have exploited the axisymmetry to rewrite the first term. We now verify that this circulation is indeed conserved. Because of the differential rotation, the contour $\mathcal{C}$ is gradually deformed in the azimuthal direction. The difference $d\phi$ in the azimuthal coordinates between two infinitesimally close fluid elements lying on $\mathcal{C}$ can be written as $d\phi = d\phi_0 + t\,d\Omega$ where $d\phi_0$ is the initial difference in azimuthal coordinates while $d\Omega$ is the difference between the elements' angular velocities. Hence we have

$$\oint_{\mathcal{C}} \Omega\varrho^2 d\phi = \oint_{\mathcal{C}} \Omega\varrho^2 d\phi_0 + t\oint_{\mathcal{C}} \Omega\varrho^2 d\Omega. \tag{2.37}$$

Note that the first term is time independent while the second one is linear in time.

The magnetic Euler equation (2.5) in cylindrical coordinate reads

$$-\Omega^2\varrho\hat{\varrho} = -\frac{\nabla p}{\rho_0} - \nabla U + \frac{\mathbf{J}\times\mathbf{B}_0}{4\pi\rho_0}. \tag{2.38}$$

Again, by the isentropic condition we can write $\nabla p/\rho_0 = \nabla w$. Taking the integral round $\mathcal{C}$ of both sides of Eq. (2.38) we have

$$-\oint_{\mathcal{C}} \Omega^2\varrho\, d\varrho = \oint_{\mathcal{C}} \frac{\mathbf{J}\times\mathbf{B}_0}{4\pi\rho_0}\cdot d\mathbf{r}. \tag{2.39}$$

Substituting from Eq. (2.39) and Eq. (2.37) into Eq. (2.36) we get

$$\begin{aligned}\Gamma &= \oint_{\mathcal{C}} \Omega\varrho^2 d\phi_0 + t\oint_{\mathcal{C}} \Omega\varrho^2 d\Omega + t\oint_{\mathcal{C}} \Omega^2\varrho\, d\varrho \\ &= \oint_{\mathcal{C}} \Omega\varrho^2 d\phi_0 + \frac{t}{2}\oint_{\mathcal{C}} d(\Omega^2\varrho^2) \\ &= \oint_{\mathcal{C}} \Omega\varrho^2 d\phi_0,\end{aligned} \tag{2.40}$$

and Γ is indeed time independent. Note that it is possible to add to $\mathbf{R}$ in Eq. (2.35) an arbitrary time independent solenoidal vector field $\mathbf{R}_0(\mathbf{r})$ which satisfies $\mathbf{R}_0\times\mathbf{v}=\nabla\chi$. However, as already stressed in the previous subsection, this will only add to Γ a time independent quantity.

It is important to note that, although the example specifically relates to an axisymmetric problem, Eq. (2.35) applies to all stationary MHD flows that have $\mathbf{J}$ collinear with $\mathbf{v}$. Accordingly, Γ will be conserved in all such flows.

III. RELATIVISTIC VARIATIONAL PRINCIPLE

In this section we formulate a Lagrangian density for MHD flow in the framework of general relativity (GR). Greek indices run from 0 to 3. The coordinates are denoted $x^\alpha=(x^0,x^1,x^2,x^3)$; x^0 stands for time. A comma denotes the usual partial derivative; a semicolon covariant differentiation. Our signature is $\{-,+,+,+\}$. We continue to take $c=1$.

A. Relativistic MHD equations

The general relativistic equations for MHD were formulated by Lichnerowicz [26], Novikov and Thorne [27], Carter [17], Bekenstein and Oron [1], and others. The role of the mass conservation equation (2.1) is taken over by the law of particle number conservation,

$$N^\alpha{}_{;\alpha} = (nu^\alpha)_{;\alpha} = 0, \tag{3.1}$$

where N^α is the particle number four-current density, n the particle proper number density, and u^α the fluid four-velocity field normalized by $u^\alpha u_\alpha = -1$. If s represents the entropy per particle (not per unit mass as in Sec. II), then the assumption of ideal adiabatic flow, Eq. (2.2), can be put in the form

$$(sN^\alpha)_{;\alpha} = 0 \quad \text{or} \quad u^\alpha s_{,\alpha} = 0. \tag{3.2}$$

Because the flow is assumed adiabatic, the energy-momentum tensor for the magnetized fluid is that of an ideal fluid augmented by the electromagnetic energy-momentum tensor:

$$T^{\alpha\beta}=pg^{\alpha\beta}+(p+\rho)u^{\alpha}u^{\beta}+(F^{\alpha\gamma}F^{\beta}{}_{\gamma}-\tfrac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g^{\alpha\beta})/(4\pi). \quad (3.3)$$

Here ρ represents the fluid's energy proper density (including rest masses) and p the scalar pressure (again assumed isotropic), while $F^{\alpha\beta}$ denotes the electromagnetic field tensor. As usual the covariant divergence $T^{\alpha\beta}{}_{;\beta}$ must vanish (energy-momentum conservation). In consequence $T^{\alpha\beta}{}_{;\beta}+u^{\alpha}u_{\gamma}T^{\gamma\beta}{}_{;\beta}=0$, which can be recast as

$$(\rho+p)u^{\beta}u^{\alpha}{}_{;\beta}=-(g^{\alpha\beta}+u^{\alpha}u^{\beta})p_{,\beta}+F^{\alpha\beta}F_{\beta}{}^{\gamma}{}_{;\gamma}/(4\pi). \quad (3.4)$$

The term $a^{\alpha}\equiv u^{\beta}u^{\alpha}{}_{;\beta}$ stands for the fluid's acceleration four-vector. The effects of gravitation are automatically included by the appeal to curved metric and covariant derivatives. This equation parallels Eq. (2.5); as usual in GR the pressure contributes alongside the energy density to the inertia. The electromagnetic field tensor is subject to Maxwell's equations

$$F^{\alpha\beta}{}_{;\beta}=4\pi j^{\alpha}, \quad (3.5)$$

$$F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0, \quad (3.6)$$

where j^{α} denotes the electric four-current density. Putting all this together we have the GR MHD Euler equation

$$(\rho+p)a^{\alpha}=-h^{\alpha\beta}p_{,\beta}+F^{\alpha\beta}j_{\beta}, \quad (3.7)$$

where we have introduced the projection tensor

$$h^{\alpha\beta}\equiv g^{\alpha\beta}+u^{\alpha}u^{\beta}. \quad (3.8)$$

The above equations do not completely specify MHD flow (as opposed to flow of a generic magnetofluid). For any flow carrying an electromagnetic field, the (antisymmetric) Faraday tensor $F_{\alpha\beta}$ may be split into electric and magnetic vectors with respect to the flow:

$$E_{\alpha}=F_{\alpha\beta}u^{\beta}, \quad (3.9)$$

$$B_{\alpha}={}^{*}F_{\beta\alpha}u^{\beta}\equiv\tfrac{1}{2}\epsilon_{\beta\alpha\gamma\delta}F^{\gamma\delta}u^{\beta}. \quad (3.10)$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Cività totally antisymmetric tensor $[\epsilon_{0123}=(-g)^{1/2}$ with g denoting the determinant of the metric $g_{\alpha\beta}]$ and ${}^{*}F_{\alpha\beta}$ is the dual of $F_{\alpha\beta}$. In the frame moving with the fluid, these four-vectors have only spatial parts which correspond to the usual **E** and **B**, respectively. The inversion of Eqs. (3.9) and (3.10) is

$$F_{\alpha\beta}=u_{\alpha}E_{\beta}-u_{\beta}E_{\alpha}+\epsilon_{\alpha\beta\gamma\delta}u^{\gamma}B^{\delta}. \quad (3.11)$$

For an infinitely conducting (perfect MHD) fluid, the electric field in the fluid's frame must vanish, i.e.,

$$E_{\alpha}=F_{\alpha\beta}u^{\beta}=0. \quad (3.12)$$

This corresponds to the usual MHD condition $\mathbf{E}+\mathbf{v}\times\mathbf{B}=0$.

B. Relativistic Lagrangian density

Inspired by Schutz's [16] Lagrangian density for pure fluids in GR, we now propose a Lagrangian density for GR MHD flow which reproduces Eqs. (3.1), (3.2), (3.5)–(3.7), and (3.12). Like Schutz we include Lin's term, which proves essential to our subsequent proof of the existence of a circulation theorem. The proposed Lagrangian density is

$$\mathcal{L}=-\rho(n,s)-F_{\alpha\beta}F^{\alpha\beta}/(16\pi)+\phi N^{\alpha}{}_{;\alpha}+\eta(sN^{\alpha})_{;\alpha}+\lambda(\gamma N^{\alpha})_{;\alpha}+\tau^{\alpha}F_{\alpha\beta}N^{\beta}. \quad (3.13)$$

Now in GR the scalar density $\mathcal{L}(-g)^{1/2}$ replaces $\mathcal{L}$ in the action (2.6), and is what enters in the Euler-Lagrange equations (2.7). The covariant derivatives cause no problem; for example, $(-g)^{1/2}\phi N^{\alpha}{}_{;\alpha}=\phi[(-g)^{1/2}N^{\alpha}]_{,\alpha}$, whose variation with respect to N^{α} is easily integrated by parts.

As in the nonrelativistic case, ϕ is the Lagrange multiplier associated with the conservation of particle number constraint, Eq. (3.1), η is that multiplier associated with the adiabatic flow constraint, Eq. (3.2), and λ is that associated with the conservation along the flow of Lin's quantity γ. We view γ, N^{α}, and s as the independent fluid variables, while n and u^{α} are determined by the obvious relations

$$-N^{\alpha}N_{\alpha}=n^{2}, \quad u^{\alpha}=n^{-1}N^{\alpha}. \quad (3.14)$$

Strictly speaking, one should include in $\mathcal{L}$ a new Lagrange multiplier times the constrained expression $N^{\alpha}N_{\alpha}+n^{2}$. Rather than clutter up $\mathcal{L}$ further, we enforce this constraint below by hand.

As usual, we view the vector potential A_{α}, rather than the electromagnetic field tensor $F_{\alpha\beta}=A_{\beta;\alpha}-A_{\alpha;\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$, as the independent electromagnetic variable. In consequence, the Maxwell Eqs. (3.6) are satisfied as identities. The last term in $\mathcal{L}$ enforces the ''vanishing of electric field'' constraint, Eq. (3.12); τ^{α} is a Lagrange multiplier four-vector field. Because here we enforce the ''vanishing of electric field'' rather than the equivalent flux freezing condition (2.4), the τ^{α} is more like **R** of Sec. II B. than like **K**. Not all of τ^{α} is physically meaningful. For suppose we add an arbitrary function $f(x^{\beta})$ multiplied by N^{α} to τ^{α}. This increments the Lagrangian density by $fN^{\alpha}F_{\alpha\beta}N^{\beta}$, which vanishes identically by the antisymmetry of $F_{\alpha\beta}$. So τ^{α} and $\tau^{\alpha}+fN^{\alpha}$ are physically equivalent. We shall exploit this to subtract from τ^{α} its component along u^{α}. So henceforth we may take it that $\tau^{\alpha}u_{\alpha}=0$.

Much freedom is still left in τ^{α}. Suppose we add to it a term proportional to $n^{-1}B^{\alpha}$. By Eqs. (3.9)–(3.11), this adds to the Lagrangian density the term $E_{\alpha}B^{\alpha}$. Of course we cannot take this to vanish at the Lagrangian level because we have not yet obtained the freezing-in condition (3.12) from it. However, it is known that $B^{\alpha}E_{\alpha}=\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$. By introducing the potential A_{α} we can write this as $\frac{1}{2}[\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}A_{\gamma}]_{;\delta}-\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta;\delta}A_{\gamma}$, where we have used the fact that $\epsilon^{\alpha\beta\gamma\delta}$ has vanishing covariant derivatives. Obviously the last term vanishes by virtue of Maxwell's equations (3.6), which are identities in the present approach. When multiplied by $(-g)^{1/2}$, the first term becomes a perfect derivative. Such a term, when added to the integral forming the Lagrangian, is known not to affect its physical content. Thus τ^{α} and $\tau^{\alpha}+\mathrm{const}\times n^{-1}B^{\alpha}$ are physically equivalent, and this transformation respects the condition $\tau_{\alpha}u^{\alpha}=0$ because $B_{\alpha}u^{\alpha}=0$ [see Eq. (3.10)]. However, there is not enough freedom in the constant to allow us to eliminate the component of τ^α along B^α. But in Sec. III E we shall exploit what we have just found.

C. Equations of motion

We can now derive the equations of motion. Variation of ϕ recovers the conservation of particles $N^\alpha{}_{;\alpha}$. Variation of λ with subsequent use of the previous result yields

$$\gamma_{,\alpha}u^\alpha=0. \tag{3.15}$$

If we vary γ we get

$$\lambda_{,\alpha}u^\alpha=0. \tag{3.16}$$

These two results are precise analogs of Eqs. (2.9) and (2.10); they inform us that γ and λ are both locally conserved with the flow. In view of the thermodynamic relation $n^{-1}(\partial\rho/\partial s)_n=T$, with T the locally measured fluid temperature, variation of s gives

$$u^\alpha\eta_{,\alpha}=-T; \tag{3.17}$$

this is the analog of Eq. (2.11).

We now vary N^α using the obvious consequence of Eq. (3.14),

$$\delta n=-u_\alpha\delta N^\alpha, \tag{3.18}$$

together with the thermodynamic relation [27] involving the specific enthalpy μ,

$$\mu\equiv(\partial\rho/\partial n)_s=n^{-1}(\rho+p); \tag{3.19}$$

we get the GR analog of Eq. (2.13),

$$\mu u_\alpha=\phi_{,\alpha}+s\,\eta_{,\alpha}+\gamma\lambda_{,\alpha}+\tau^\beta F_{\alpha\beta}. \tag{3.20}$$

The importance of Lin's γ is again clear here; in the pure isentropic fluid case ($F^{\alpha\beta}=0$ and $s=\text{const}$), the Khalatnikov vorticity tensor given by

$$\omega_{\alpha\beta}=(\mu u_\beta)_{,\alpha}-(\mu u_\alpha)_{,\beta}=(\gamma\lambda_{,\beta})_{,\alpha}-(\gamma\lambda_{,\alpha})_{,\beta} \tag{3.21}$$

would vanish in the absence of γ, thus constraining us to discuss only irrotational flow.

By contracting Eq. (3.20) with u^α and using $u_\alpha u^\alpha=-1$ as well as Eqs. (3.12) and (3.16), (3.17), we get the following GR version of Eq. (2.12):

$$\phi_{,\alpha}u^\alpha=-\mu+Ts. \tag{3.22}$$

Thus the proper time rate of change of ϕ along the flow is just minus the specific Gibbs energy or chemical potential. The apparent difference between the result here and Eq. (2.12) stems from the fact that proper time rate (here) and coordinate time rate (there) differ by gravitational redshift and time dilation effects. These effects are not noticeable when one compares Eq. (3.17) with (2.11) because the first refers to locally measured temperature and the second to global temperature; these two temperatures differ by the same factors as do proper and coordinate time.

Turn now to the variation of A_α. Because of the antisymmetry of $F_{\alpha\beta}$, the last term of the Lagrangian Eq. (3.13) can be written as $(\tau^\beta N^\alpha-\tau^\alpha N^\beta)A_{\alpha,\beta}$. The variation of A_α in the corresponding term in the action produces, after integration by parts, the term $[(-g)^{1/2}(\tau^\alpha N^\beta-\tau^\beta N^\alpha)]_{,\beta}\delta A_\alpha$. Because for any antisymmetric tensor $t^{\alpha\beta}$, $(-g)^{1/2}t^{\alpha\beta}{}_{;\beta}=[(-g)^{1/2}t^{\alpha\beta}]_{,\beta}$, this finally leads to the equation

$$F^{\alpha\beta}{}_{;\beta}=4\pi(\tau^\alpha N^\beta-\tau^\beta N^\alpha)_{;\beta}. \tag{3.23}$$

Comparison with Eq. (3.5) shows that this result gives us a representation of the electric current density j^α as the divergence of the bivector $\tau^\alpha N^\beta-\tau^\beta N^\alpha$. Such a representation makes the conservation of charge ($j^\alpha{}_{;\alpha}=0$) an identity because the divergence of the divergence of any antisymmetric tensor vanishes. This equation is the GR analog of Eq. (2.17). Formally, Eq. (3.23) determines the Lagrange multiplier four-vector τ^α, modulo the freedom inherent in it.

D. MHD Euler equation in general relativity

Our central task now is to show that the equations in Sec. III C lead uniquely to the GR MHD Euler equation (3.7). We begin by writing the Khalatnikov vorticity $\omega_{\beta\alpha}$ in two forms,

$$\omega_{\beta\alpha}=\mu_{,\beta}u_\alpha-\mu_{,\alpha}u_\beta+\mu u_{\alpha;\beta}-\mu u_{\beta;\alpha}, \tag{3.24}$$

as well as by means of Eq. (3.20),

$$\omega_{\beta\alpha}=s_{,\beta}\eta_{,\alpha}-s_{,\alpha}\eta_{,\beta}+\gamma_{,\beta}\lambda_{,\alpha}-\gamma_{,\alpha}\lambda_{,\beta}+\tau^\delta{}_{;\beta}F_{\alpha\delta}-\tau^\delta{}_{;\alpha}F_{\beta\delta}+\tau^\delta F_{\alpha\delta;\beta}-\tau^\delta F_{\beta\delta;\alpha}. \tag{3.25}$$

Contracting the left hand side of the first with N^α, recalling Eq. (3.14), and that by normalization $u^\alpha u_{\alpha;\beta}=0$ whereas $u^\beta u_{\alpha;\beta}=a_\alpha$, the fluid's four-acceleration, we get

$$\omega_{\beta\alpha}N^\alpha=-n\mu_{,\beta}-n\mu_{,\alpha}u^\alpha u_\beta-n\mu a_\beta=-nh_\beta{}^\alpha\mu_{,\alpha}-n\mu a_\beta. \tag{3.26}$$

Now contracting Eq. (3.25) with N^α and using Eqs. (3.15)–(3.17) and (3.12) to drop a number of terms we get

$$\omega_{\beta\alpha}N^\alpha=-nTs_{,\beta}-\tau^\delta{}_{;\alpha}F_{\beta\delta}N^\alpha+\tau^\delta F_{\alpha\delta;\beta}N^\alpha-\tau^\delta F_{\beta\delta;\alpha}N^\alpha. \tag{3.27}$$

By virtue of Eq. (3.2), $-nTs_{,\beta}$ is the same as $-nTh_\beta{}^\alpha s_{,\alpha}$. It is convenient to use the thermodynamic identity $d\mu=n^{-1}dp+Tds$, which follows from Eq. (3.19), and the first law $d(\rho/n)=Tds-pd(1/n)$, to replace in Eq. (3.27) $-nTs_{,\beta}$ by $h_\beta{}^\alpha(-n\mu_{,\alpha}+p_{,\alpha})$. Equating our two expressions for $\omega_{\beta\alpha}N^\alpha$ gives, after a cancellation,

$$-(n\mu a_\beta+h_\beta{}^\alpha p_{,\alpha})=-\tau^\delta{}_{;\alpha}F_{\beta\delta}N^\alpha+\tau^\delta F_{\alpha\delta;\beta}N^\alpha-\tau^\delta F_{\beta\delta;\alpha}N^\alpha. \tag{3.28}$$

The last two terms in this equation can be combined into a single one by virtue of Eq. (3.6), which, as well known, can be written with covariant as well as ordinary derivatives. Further, by Eq. (3.19) we may replace $n\mu$ by $\rho+p$. In this manner we get

$$(\rho+p)a_\beta=-h_\beta{}^\alpha p_{,\alpha}+F_{\beta\alpha;\delta}\tau^\delta N^\alpha+F_{\beta\delta}\tau^\delta{}_{;\alpha}N^\alpha. \qquad (3.29)$$

The term $\tau^\delta{}_{;\alpha}N^\alpha$ here can be replaced by two others with the help of Eq. (3.23) if we take into account that $N^\beta{}_{;\beta}=0$:

$$(\rho+p)a_\beta=-h_\beta{}^\alpha p_{,\alpha}+F_{\beta\delta}F^{\delta\alpha}{}_{;\alpha}/(4\pi)$$
$$+F_{\beta\alpha;\delta}\tau^\delta N^\alpha+F_{\beta\delta}(\tau^\alpha N^\delta)_{;\alpha}. \qquad (3.30)$$

We note that the last two terms on the right hand side combine into $(F_{\beta\alpha}N^\alpha\tau^\delta)_{;\delta}$ which vanishes by Eq. (3.12). Now substituting from the Maxwell equations (3.5) we arrive at the final equation

$$(\rho+p)a_\beta=-h_\beta{}^\alpha p_{,\alpha}+F_{\beta\delta}j^\delta, \qquad (3.31)$$

which is the correct GR MHD Euler equation (3.7). We have not used any information about τ^α beyond Eq. (3.23); hence Euler's equation is valid for all choices of τ^α. Since we are able to obtain all equations of motion for GR MHD from our Lagrangian density, we may regard it as correct, and go on to look at some consequences.

E. New circulation conservation law

Equations (3.20) and (3.15), (3.16) allow us to generalize the conserved circulation of Sec. II D to relativistic perfect MHD. Let Γ be the line integral

$$\Gamma=\oint_{\mathcal{C}} z_\alpha dx^\alpha, \qquad (3.32)$$

where $\mathcal{C}$ is, again, a closed curve drifting with the fluid, and

$$z_\alpha\equiv\mu u_\alpha-\tau^\beta F_{\alpha\beta}. \qquad (3.33)$$

According to Eq. (3.20), $z_\alpha=\phi_{,\alpha}+s\,\eta_{,\alpha}+\gamma\lambda_{,\alpha}$. Since $\phi_{,\alpha}$ is a gradient, its contribution to Γ vanishes. Likewise, for isentropic flow ($s=$const) the term involving $s\,\eta_{,\alpha}$ makes no contribution to Γ. Thus

$$\Gamma=\oint_{\mathcal{C}}\gamma\lambda_{,\alpha}dx^\alpha=\oint_{\mathcal{C}}\gamma d\lambda. \qquad (3.34)$$

By Eqs. (3.15) and (3.16) both γ and λ are conserved with the flow. Thus Γ is conserved along the flow. Note that by virtue of γ's presence Γ need not vanish.

In the absence of electromagnetic fields and in the nonrelativistic limit ($\mu\rightarrow m$ where m is a fluid particle's rest mass), Γ for a curve $\mathcal{C}$ taken at constant time reduces to Kelvin's circulation. On this ground our result can be considered a generalization of Kelvin's circulation theorem to general relativistic MHD. We have gone here beyond Bekenstein and Oron's original result [1] in that no symmetry is necessary for the circulation to be conserved.

To manifestly exhibit the conserved circulation, one has to know τ^α explicitly. The first step is to understand the freedom left in τ^α beyond that discussed in Sec. III B. The second is to determine τ^α in a specific flow, exploiting for this the symmetries and other information. Below we address the first step; the second is left mainly to future publications.

Given a specific MHD flow as background, let us define a generic test field $f_{\alpha\beta}=-f_{\beta\alpha}$ that satisfies Maxwell's homogeneous equations (3.6) as well as the freezing-in condition (3.12), e.g., $e_\alpha\equiv f_{\alpha\beta}u^\beta=0$. We think of $f_{\alpha\beta}$ as very weak, so that it does not disturb the MHD flow or the spacetime geometry; it is a passive tensor. Because $f_{\alpha\beta}u^\beta=0$, $f_{\alpha\beta}$ has only three independent components. Therefore, its full content is reflected in the ''magnetic four-vector'' $b_\alpha\equiv\frac{1}{2}\epsilon_{\beta\alpha\gamma\delta}f^{\gamma\delta}u^\beta$, which is obviously orthogonal to u_α. The transformation $\tau^\alpha\rightarrow\tau^\alpha+kn^{-1}b^\alpha$ (k a real constant) is not a symmetry of the Lagrangian. However, it does not disturb the inhomogeneous Maxwell equations (3.5) and (3.23). This is because the change in τ^α merely adds to the electric current the term $(b^\alpha u^\beta-b^\beta u^\alpha)_{;\beta}=(-g)^{-1/2}[(-g)^{1/2}(b^\alpha u^\beta-b^\beta u^\alpha)]_{,\beta}$. Because of the condition $e^\alpha=0$, we may easily invert the analog of Eq. (3.11) to get $b^\alpha u^\beta-b^\beta u^\alpha=\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}f_{\gamma\delta}$. But, since $(-g)^{1/2}\epsilon^{\alpha\beta\gamma\delta}$ is just the constant antisymmetric symbol, our assumed equations $f_{\alpha\beta,\gamma}+f_{\gamma\alpha,\beta}+f_{\beta\gamma,\alpha}=0$ imply that $(b^\alpha u^\beta-b^\beta u^\alpha)_{;\beta}=0$, so that the Maxwell equations (3.23) are invariant under $\tau^\alpha\rightarrow\tau^\alpha+kn^{-1}b^\alpha$. So is the Euler equation, since its derivation used only the information about τ^α inherent in Eq. (3.23).

The expression for u_α, Eq. (3.20), does seem to change under $\tau^\alpha\rightarrow\tau^\alpha+kn^{-1}b^\alpha$, and we also note that $\Gamma\rightarrow\Gamma+k\oint n^{-1}b^\beta F_{\alpha\beta}dx^\alpha$. Now since the ''magnetic four-vector'' b^α is frozen in, like any such *infinitesimal* field, it evolves in such a way that $n^{-1}b^\alpha$ gives for all time that part of the spacetime separation of two neighboring fluid elements that is orthogonal to u^α [1]; cf. the discussion after Eq. (2.32). Thus $n^{-1}b^\alpha$ can be employed to define a thin closed strip dragged with the fluid such that one of its edges coincides with the curve $\mathcal{C}$. Therefore, the increment $\oint n^{-1}b^\beta F_{\alpha\beta}dx^\alpha$ is just the conserved magnetic flux through this strip. Evidently the transformation $\tau^\alpha\rightarrow\tau^\alpha+kn^{-1}b^\alpha$ has not changed the nature of the conservation law for Γ, but only added a trivially conserved quantity to it.

Now the MHD flow $\{B^\alpha,u^\alpha,n,\rho,\mu\}$ is evidently unchanged because neither the MHD Euler equation (3.4) nor Maxwell's equations were changed, so we must conclude that, in the expression for u^α, Eq. (3.20), the change of the $\tau^\beta F_{\alpha\beta}$ term must be compensated by suitable changes in the pair of Lagrange multipliers $\phi+s\,\eta$ and λ (since we are assuming $s=$const). They are capable of this because b^α has only two independent components, for the condition $b^\alpha u_\alpha=0$ eliminates one of the four. In addition b^α comes from $f_{\alpha\beta}$ which satisfies Eqs. (3.6); in particular, $f_{12,3}+f_{31,2}+f_{23,1}=0$ in the chosen coordinates. But since no time derivatives appear in it, this last equation serves as an initial constraint on b^α just as the Gauss equation $\nabla\cdot\mathbf{B}=0$ does for the true magnetic field. Accordingly, one further relation exists between components of b^α so that the generic b^α contains only two free functions. Thus the change in $\tau^\beta F_{\alpha\beta}$ can be taken up by changes in the two functions $\phi+s\,\eta$ and λ so that μu_α is unchanged.

Note that it is not possible to ''get rid'' of τ^α by means of the transformation $\tau^\alpha\rightarrow\tau^\alpha+kn^{-1}b^\alpha$ because, as we shall make clear presently, τ^α and b^α obey different equations of motion. Thus there must be a residual part of τ^α that is not changed by the transformations. It is this part that is responsible for the conserved circulation, so that the conservation of Γ cannot be reduced to magnetic flux conservation.

The following algorithm can be used to find τ^α. Maxwell's inhomogeneous equations (3.23), which say that the divergence of a certain tensor vanishes, can always be ''solved'' by the prescription

$$F^{\alpha\beta}-4\pi(\tau^\alpha N^\beta-\tau^\beta N^\alpha)=\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\gamma\delta}, \tag{3.35}$$

where the new field $\mathcal{F}_{\gamma\delta}$ just has to satisfy Maxwell's homogeneous equations (3.6), i.e., $\mathcal{F}_{\gamma\delta}\equiv\mathcal{A}_{\delta,\gamma}-\mathcal{A}_{\gamma,\delta}$. Taking the dual of Eq. (3.35) with the help of the identity $\epsilon_{\gamma\delta\alpha\beta}\epsilon^{\alpha\beta\mu\nu}=-2(\delta_\gamma{}^\mu\delta_\delta{}^\nu-\delta_\gamma{}^\nu\delta_\delta{}^\mu)$ gives

$$^*F_{\gamma\delta}-4\pi\epsilon_{\gamma\delta\alpha\beta}\tau^\alpha N^\beta=-\mathcal{F}_{\gamma\delta}. \tag{3.36}$$

Contracting this equation with u^γ gives the further requirement on $\mathcal{F}_{\alpha\beta}$:

$$\mathcal{F}_{\delta\gamma}u^\gamma=B_\delta, \tag{3.37}$$

where we have used Eq. (3.10). The $\mathcal{F}_{\delta\gamma}$ can always be solved for: because of gauge freedom there are three independent components in $\mathcal{A}_\alpha$, and this is enough to find a solution for an arbitrary field B_δ obeying $B_\alpha u^\alpha=0$ (thus three components at most). If fact, B_δ does not determine $\mathcal{F}_{\delta\gamma}$ uniquely: if one adds to this last one of the frozen-in $f_{\gamma\delta}$ we discussed earlier in this section (which also satisfy the homogeneous Maxwell equations), Eq. (3.37) is still satisfied because $f_{\delta\gamma}u^\gamma=0$.

We get τ^α by contracting Eq. (3.35) by u_β and remembering that $F^{\alpha\beta}u_\beta=0$ and $\tau^\beta u_\beta=0$. Thus

$$\tau^\alpha=(8\pi n)^{-1}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\gamma\delta}u_\beta. \tag{3.38}$$

It is interesting that B_δ plays the role of the electric part of $\mathcal{F}_{\delta\gamma}$ while τ^α enters like the magnetic part of this tensor; cf. Eq. (3.10) (but because $\mathcal{F}_{\delta\gamma}u^\gamma\neq 0$, τ^α evolves differently from a magnetic type field like B^α or the b^α). It should also be clear now that the freedom in redefining $\mathcal{F}_{\gamma\delta}\to\mathcal{F}_{\gamma\delta}+f_{\gamma\delta}$ is equivalent to the changes $\tau^\alpha\to\tau^\alpha+kn^{-1}b^\alpha$ we considered earlier in this section. This freedom can be exploited together with the symmetries to simplify the problem of solving explicitly for τ^α in any specific MHD flow.

[1] J.D. Bekenstein and E. Oron, Phys. Rev. D **18**, 1809 (1978).
[2] J.D. Bekenstein, Astrophys. J. **319**, 207 (1987).
[3] J.W. Herivel, Proc. Cambridge Philos. Soc. **51**, 344 (1955).
[4] C. Eckart, Phys. Fluids **3**, 421 (1960).
[5] C.C. Lin, in *Liquid Helium*, Proceedings of the International School of Physics ''Enrico Fermi,'' Course XXI, edited by G. Careri (Academic Press, New York, 1963).
[6] R.L. Seliger and F.R.S. Witham, Proc. R. Soc. London, Ser. A **305**, 1 (1968).
[7] L. Mittag, M. J. Stephen, and W. Yourgrau, in *Variational Principles in Dynamics and Quantum Theory*, edited by W. Yourgrau and S. Mandelstam (Dover, New York, 1968).
[8] C. Eckart, Phys. Rev. **54**, 920 (1938).
[9] F.S. Henyey, Phys. Rev. A **26**, 480 (1982).
[10] W.A. Newcomb, Nucl. Fusion Suppl. **2**, 451 (1962).
[11] T.S. Lundgren, Phys. Fluids **6**, 1313 (1963).
[12] P. Penfield, Jr., Phys. Fluids **9**, 1184 (1966).
[13] A.H. Taub, Phys. Rev. **94**, 1468 (1954).
[14] A.H. Taub, Commun. Math. Phys. **15**, 235 (1969).
[15] T. Kodama, Th.-H. Elze, Y. Hama, M. Makler, and J. Rafelski, J. Phys. G **25**, 1935 (1999).
[16] B.F. Schutz, Phys. Rev. D **2**, 2762 (1970).
[17] B. Carter, in *Active Galactic Nuclei*, edited by C. Hazard and S. Mitton (Cambridge University Press, Cambridge, 1977).
[18] B. Carter, in *Topological Defects and Nonequilibrium Dynamics of Symmetry Breaking Phase Transitions*, 1999 Les Houches Lectures, edited by Y. Bunkov and H. Godfrin (Kluwer, Dordrecht, 2000).
[19] A. Achterberg, Phys. Rev. A **28**, 2449 (1983).
[20] C. Thompson, Phys. Rev. D **57**, 3219 (1998).
[21] J.S. Heyl and L. Hernquist, Phys. Rev. D **59**, 045 005 (1999).
[22] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon, Oxford, 1984), p. 227.
[23] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Instability* (Clarendon Press, Oxford, 1961).
[24] L. Mestel, *Stellar Magnetism* (Clarendon Press, Oxford, 1999).
[25] V.C.A. Ferraro, Mon. Not. R. Astron. Soc. **97**, 458 (1937).
[26] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967).
[27] I.D. Novikov and K.S. Thorne, in *Black Holes*, edited by B.S. DeWitt and C.M. DeWitt (Gordon and Breach, New York, 1973).